Lecture 11

Pole Placement

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Pole Placement

- In this lecture we will discuss a design method commonly called the \textit{pole-placement} or \textit{pole-assignment technique}.

- We assume that all state variables are measurable and are available for feedback.

- If the system considered is completely state controllable, then poles of the closed-loop system may be placed at any desired locations by means of state feedback through an appropriate state feedback gain matrix.
Pole Placement

• The present design technique begins with a determination of the desired closed-loop poles based on the transient-response and/or frequency-response requirements, such as speed, damping ratio, or bandwidth, as well as steady-state requirements.

• By choosing an appropriate gain matrix for state feedback, it is possible to force the system to have closed-loop poles at the desired locations, provided that the original system is completely state controllable.
Topology of Pole Placement

- Consider a plant represented in state space by

\[
\dot{x} = Ax + Bu \\
y = Cx
\]
Topology of Pole Placement

• In a typical feedback control system, the output, $y$, is fed back to the summing junction.

• It is now that the topology of the design changes. Instead of feeding back $y$, we feed back all of the state variables.

• If each state variable is fed back to the control, $u$, through a gain, $k_i$, there would be $n$ gains, $k_i$, that could be adjusted to yield the required closed-loop pole values.
The feedback through the gains, $k_i$, is represented in the following figure by the feedback vector $K$.

$$\dot{x} = Ax + B(r - Kx)$$

$$\dot{x} = Ax + Br - BKx$$

$$\dot{x} = (A - BK)x + Br$$

$$y = Cx$$
Pole Placement

- We will limit our discussions to single-input, single-output systems (i.e. we will assume that the control signal $u(t)$ and output signal $y(t)$ to be scalars).
- We will also assume that the reference input $r(t)$ is zero.

\[
\dot{x} = (A - BK)x + Br \\
\dot{x} = (A - BK)x \quad u = -Kx
\]
Pole Placement

\[ \dot{x} = (A - BK)x \]

- The stability and transient response characteristics are determined by the eigenvalues of matrix \( A - BK \).

- If matrix \( K \) is chosen properly, Eigenvalues of the system can be placed at desired location.

- And the problem of placing the regulator poles (closed-loop poles) at the desired location is called a pole-placement problem.
Pole Placement

- There are three approaches that can be used to determine the gain matrix $K$ to place the poles at desired location.
  
  - Direct Substitution Method.
  
  - Ackermann’s formula.
  
  - Using Transformation Matrix $P$.

- All those method yields the same result.
Direct Substitution Method
Pole Placement (Direct Substitution Method)

- Following are the steps to be followed in this particular method.

1. Check the state controllability of the system

\[ CM = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B] \]
Pole Placement (Direct Substitution Method)

- **Steps:**

1. Check the state controllability of the system.
   
   \[ C_T = [B \ AB \ A^2B \ ... \ A^{n-1}B] \]

2. Define the state feedback gain matrix as
   
   \[ K = [k_1 \ k_2 \ k_3\cdots \ k_n] \]

   - And equating \(|sI - A + BK|\) with desired characteristic equation.

\[(s - \mu_1)(s - \mu_2)\cdots(s - \mu_n) = s^n + \alpha_1s^{n-1} + \alpha_2s^{n-2} + \cdots + \alpha_{n-1}s + \alpha_n\]
Example

- Consider the regulator system shown in following figure. The plant is given by

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} = \begin{bmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  -1 & -5 & -6
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} + \begin{bmatrix}
  0 \\
  0 \\
  1
\end{bmatrix} u(t)
\]

- The system uses the state feedback control \( u = -Kx \). The desired eigenvalues are \( \mu_1 = -2 + j4, \mu_2 = -2 - j4, \mu_3 = -1 \). Determine the state feedback gain matrix \( K \).
Example

- **Step-1**

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} = 
\begin{bmatrix}
    0 & 1 & 0 \\
    0 & 0 & 1 \\
    -1 & -5 & -6
\end{bmatrix} 
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} + 
\begin{bmatrix}
    0 \\
    0 \\
    1
\end{bmatrix} u(t)
\]

- First, we need to check the controllability matrix of the system. Since the controllability matrix \( C_T \) is given by

\[
C_T = [B \quad AB \quad A^2B] = 
\begin{bmatrix}
    0 & 0 & 1 \\
    0 & 1 & -6 \\
    1 & -6 & 31
\end{bmatrix}
\]

- We find that \( \text{rank}(C_T)=3 \). Thus, the system is completely state controllable and arbitrary pole placement is possible.
Step-2:

Let $K$ be

$$K = [k_1 \ k_2 \ k_3]$$

$$|sI - A + BK| = \begin{vmatrix}
    s & 0 & 0 \\
    0 & s & 0 \\
    0 & 0 & s
\end{vmatrix} - \begin{vmatrix}
    0 & 1 & 0 \\
    0 & 0 & 1 \\
    -1 & -5 & -6
\end{vmatrix} + \begin{vmatrix}
    0
\end{vmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

$$= s^3 + (6 + k_3)s^2 + (5 + k_2)s + 1 + k_1$$

Desired characteristic polynomial is obtained as

$$(s + 2 - 4j)(s + 2 + 4j)(s + 10) = s^3 + 14s^2 + 60s + 200$$

Comparing the coefficients of powers of $s$

$$14 = (6 + k_3) \quad k_3 = 8$$

$$60 = (5 + k_2) \quad k_2 = 55$$

$$200 = 1 + k_1 \quad k_1 = 199$$
Ackermann’s Formula
Pole Placement (Ackermann’s Formula)

• Following are the steps to be followed in this particular method.

1. Check the state controllability of the system

\[ CM = [B \ AB \ A^2B \ \ldots \ A^{n-1}B] \]
Pole Placement (Ackermann’s Formula)

• Following are the steps to be followed in this particular method.

2. Use Ackermann’s formula to calculate $K$

$$K = [0 \ 0 \ \cdots \ 0 \ 1][B \ AB \ A^2B \ \cdots \ A^{n-1}B]^{-1}\phi(A)$$

$$\phi(A) = A^n + \alpha_1A^{n-1} + \cdots + \alpha_{n-1}A + \alpha_nI$$
Pole Placement (Ackermann’s Formula)

**Example-1**: Consider the regulator system shown in following figure. The plant is given by

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} =
\begin{bmatrix}
    0 & 1 & 0 \\
    0 & 0 & 1 \\
    -1 & -5 & -6
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} +
\begin{bmatrix}
    0 \\
    0 \\
    1
\end{bmatrix} u(t)
\]

The system uses the state feedback control \( u = -Kx \). The desired eigenvalues are \( \mu_1 = -2 + j4, \mu_2 = -2 - j4, \mu_3 = -1 \). Determine the state feedback gain matrix \( K \).
Pole Placement (Using Transformation Matrix $P$)

- **Example-1: Step-1**

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
\end{bmatrix} = 
\begin{bmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  -1 & -5 & -6 \\
\end{bmatrix} 
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
\end{bmatrix} + 
\begin{bmatrix}
  0 \\
  0 \\
  1 \\
\end{bmatrix} u(t)
\]

- First, we need to check the controllability matrix of the system. Since the controllability matrix $CM$ is given by

\[
CM = [B \quad AB \quad A^2B] = 
\begin{bmatrix}
  0 & 0 & 1 \\
  0 & 1 & -6 \\
  1 & -6 & 31 \\
\end{bmatrix}
\]

- We find that $\text{rank}(CM)=3$. Thus, the system is completely state controllable and arbitrary pole placement is possible.
Pole Placement (Ackermann’s Formula)

- Following are the steps to be followed in this particular method.

2. Use Ackermann’s formula to calculate $K$

$$ K = [0 \ 0 \ 1][B \ AB \ A^2B]^{-1}\phi(A) $$

$$ \phi(A) = A^3 + \alpha_1 A^2 + \alpha_2 A + \alpha_3 I $$

- $\alpha_i$ are the coefficients of the desired characteristic polynomial.

$$(s + 2 - 4j)(s + 2 + 4j)(s + 10) = s^3 + 14s^2 + 60s + 200$$

$$\alpha_1 = 14, \quad \alpha_2 = 60, \quad \alpha_3 = 200$$
Pole Placement (Ackermann’s Formula)

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} = 
\begin{bmatrix}
    0 & 1 & 0 \\
    0 & 0 & 1 \\
    -1 & -5 & -6
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} + 
\begin{bmatrix}
    0 \\
    0 \\
    1
\end{bmatrix} u(t)
\]

\[
\phi(A) = A^3 + 14A^2 + 60A + 200I
\]

\[
\phi(A) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^3 + 14 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^2 + 60 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + 200 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\phi(A) = \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -34 & 117 \end{bmatrix}
\]
Pole Placement (Ackermann’s Formula)

\[
\begin{bmatrix}
B & AB & A^2B
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & -6 \\
1 & -6 & 31
\end{bmatrix}
\]

\[
\emptyset(A) = \begin{bmatrix}
199 & 55 & 8 \\
-8 & 159 & 7 \\
-7 & -34 & 117
\end{bmatrix}
\]

\[
K = [0 \ 0 \ 1][B \ AB \ A^2]^{-1}\emptyset(A)
\]

\[
K = [0 \ 0 \ 1]\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & -6 \\
1 & -6 & 31
\end{bmatrix}^{-1}\begin{bmatrix}
199 & 55 & 8 \\
-8 & 159 & 7 \\
-7 & -34 & 117
\end{bmatrix}
\]

\[
K = [199 \ 55 \ 8]
\]
Pole Placement

• **Example-2:** Consider the regulator system shown in following figure. The plant is given by

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t)
\]

• Determine the state feedback gain for each state variable to place the poles at \(-1+j, -1-j, -3\). (Apply all methods)
End of Lec