Chapter 32

Inductance

Joseph Henry

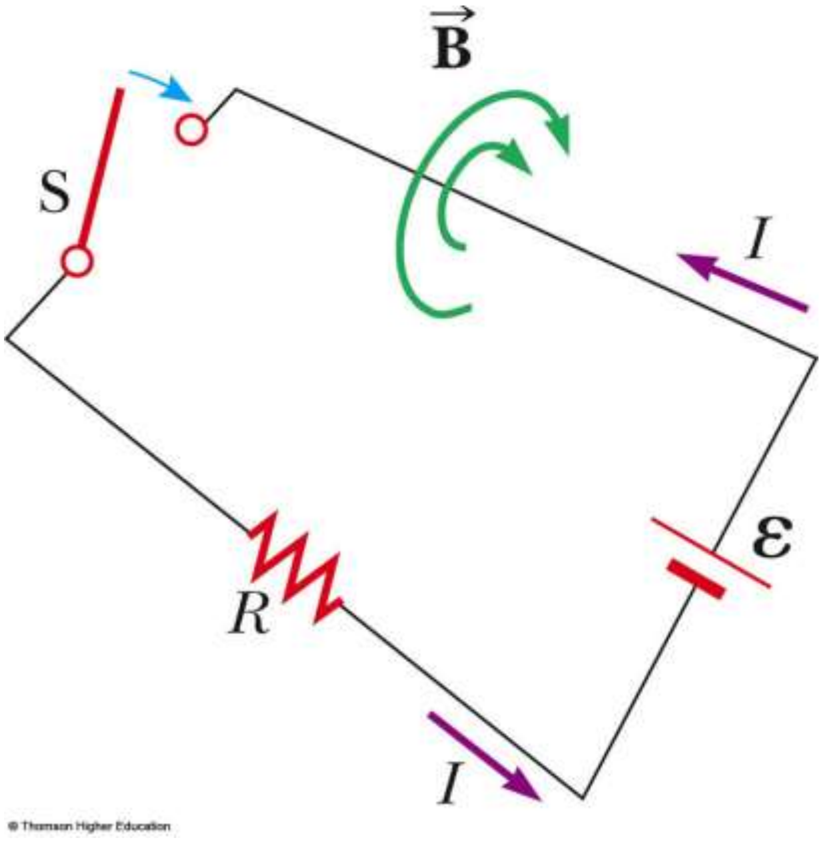
- 1797 – 1878
- American physicist
- First director of the Smithsonian
- Improved design of electromagnet
- Constructed one of the first motors
- Discovered self-inductance
- Unit of inductance is named in his honor

Some Terminology

- Use *emf* and *current* when they are caused by batteries or other sources
- Use *induced emf* and *induced current* when they are caused by changing magnetic fields
- When dealing with problems in electromagnetism, it is important to distinguish between the two situations

Self-Inductance

- When the switch is closed, the current does not immediately reach its maximum value
- Faraday's law can be used to describe the effect



Self-Inductance, 2

- As the current increases with time, the magnetic flux through the circuit loop due to this current also increases with time
- This increasing flux creates an induced emf in the circuit

Self-Inductance, 3

- The direction of the induced emf is such that it would cause an induced current in the loop which would establish a magnetic field opposing the change in the original magnetic field
- The direction of the induced emf is opposite the direction of the emf of the battery
- This results in a gradual increase in the current to its final equilibrium value

Self-Inductance, 4

- This effect is called **self-inductance**
  - Because the changing flux through the circuit and the resultant induced emf arise from the circuit itself
- The emf ε_L is called a **self-induced emf**

Self-Inductance, Equations

- An induced emf is always proportional to the time rate of change of the current
  - The emf is proportional to the flux, which is proportional to the field and the field is proportional to the current

$$\varepsilon_L = -L\frac{dI}{dt}$$

- L is a constant of proportionality called the **inductance** of the coil and it depends on the geometry of the coil and other physical characteristics

Inductance of a Coil

- A closely spaced coil of N turns carrying current *I* has an inductance of

$$L = \frac{N\Phi_B}{I} = -\frac{\varepsilon_L}{dI/dt}$$

- The inductance is a measure of the opposition to a change in current

Inductance Units

- The SI unit of inductance is the **henry** (H)

$$1\text{H} = 1\frac{\text{V}\cdot\text{s}}{\text{A}}$$

- Named for Joseph Henry

Inductance of a Solenoid

- Assume a uniformly wound solenoid having N turns and length ℓ
  - Assume ℓ is much greater than the radius of the solenoid
- The flux through each turn of area A is

$$\Phi_B = BA = \mu_o n I A = \mu_o \frac{N}{\ell} I A$$

Inductance of a Solenoid, cont

- The inductance is

$$L = \frac{N\Phi_B}{I} = \frac{\mu_o N^2 A}{\ell}$$

- This shows that L depends on the geometry of the object

RL Circuit, Introduction

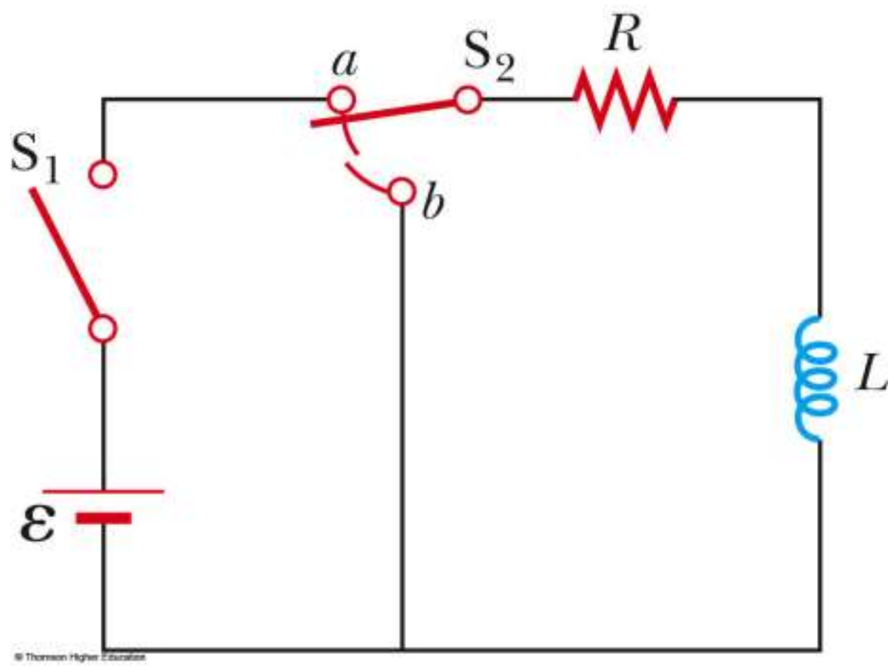
- A circuit element that has a large self-inductance is called an **inductor**
- The circuit symbol is 
- We assume the self-inductance of the rest of the circuit is negligible compared to the inductor
  - However, even without a coil, a circuit will have some self-inductance

Effect of an Inductor in a Circuit

- The inductance results in a back emf
- Therefore, the inductor in a circuit opposes changes in current in that circuit
  - The inductor attempts to keep the current the same way it was before the change occurred
  - The inductor can cause the circuit to be "sluggish" as it reacts to changes in the voltage

RL Circuit, Analysis

- An *RL* circuit contains an inductor and a resistor
- Assume S_2 is connected to a
- When switch S_1 is closed (at time $t = 0$), the current begins to increase
- At the same time, a back emf is induced in the inductor that opposes the original increasing current



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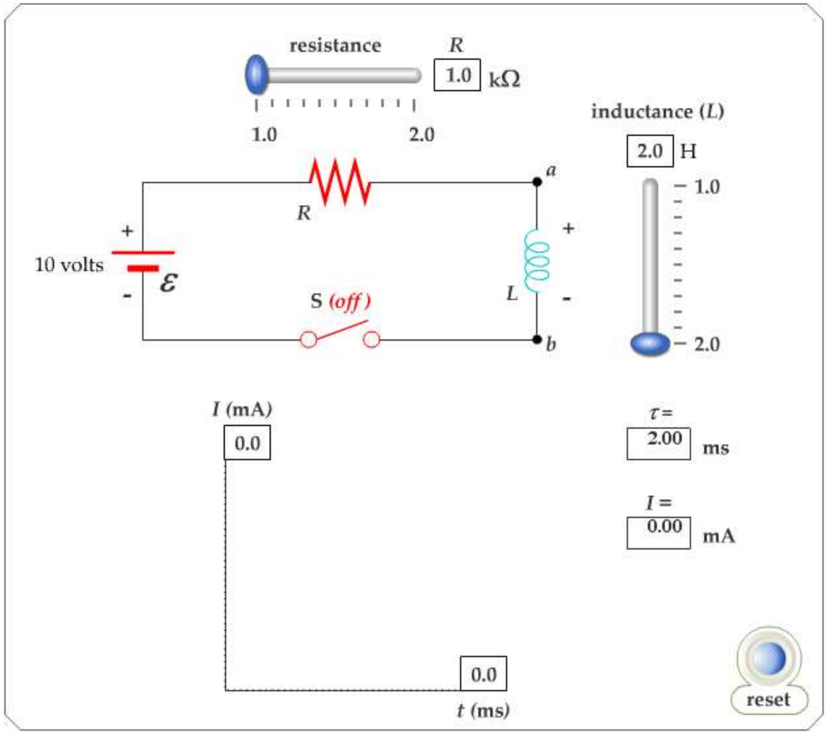
ACTIVE FIGURE

Active Figure 32.2 (a)

Use the active figure to set R and L and see the effect on the current



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RL Circuit, Analysis, cont.

- Applying Kirchhoff's loop rule to the previous circuit in the clockwise direction gives

$$\varepsilon - IR - L\frac{dI}{dt} = 0$$

- Looking at the current, we find

$$I = \frac{\varepsilon}{R}\left(1 - e^{-Rt/L}\right)$$

RL Circuit, Analysis, Final

- The inductor affects the current exponentially
- The current does not instantly increase to its final equilibrium value
- If there is no inductor, the exponential term goes to zero and the current would instantaneously reach its maximum value as expected

RL Circuit, Time Constant

- The expression for the current can also be expressed in terms of the time constant, τ, of the circuit

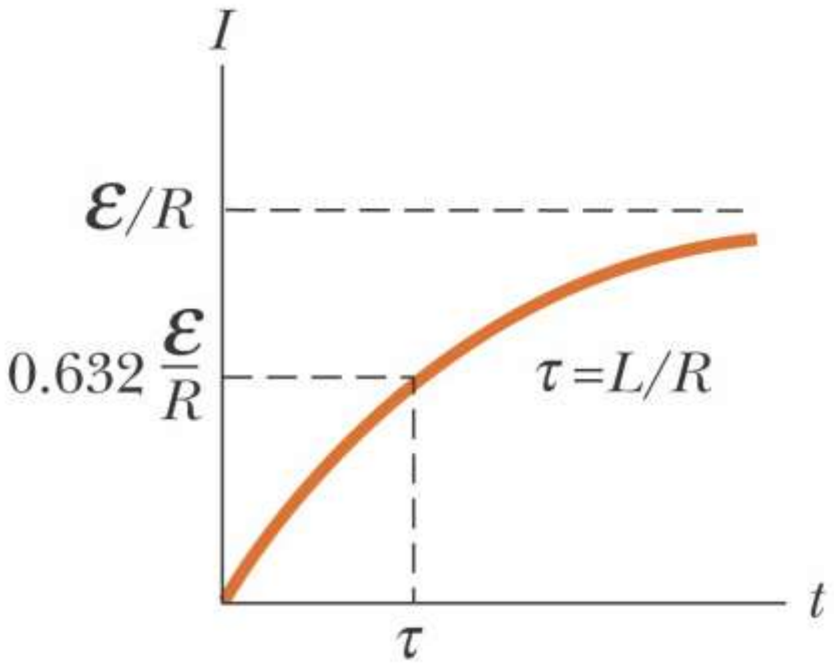
$$I = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau}\right)$$

  - where $\tau = L / R$

- Physically, τ is the time required for the current to reach 63.2% of its maximum value

RL Circuit, Current-Time Graph, (1)

- The equilibrium value of the current is ε / R and is reached as t approaches infinity
- The current initially increases very rapidly
- The current then gradually approaches the equilibrium value
- Use the active figure to watch the graph



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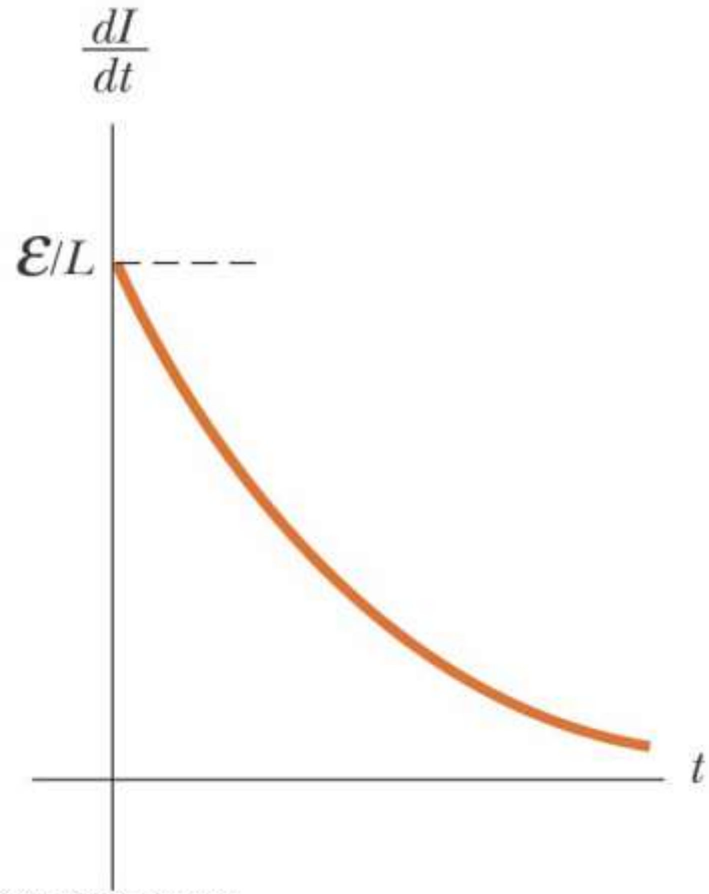
RL Circuit, Current-Time Graph, (2)

- The time rate of change of the current is a maximum at $t = 0$
- It falls off exponentially as t approaches infinity
- In general,

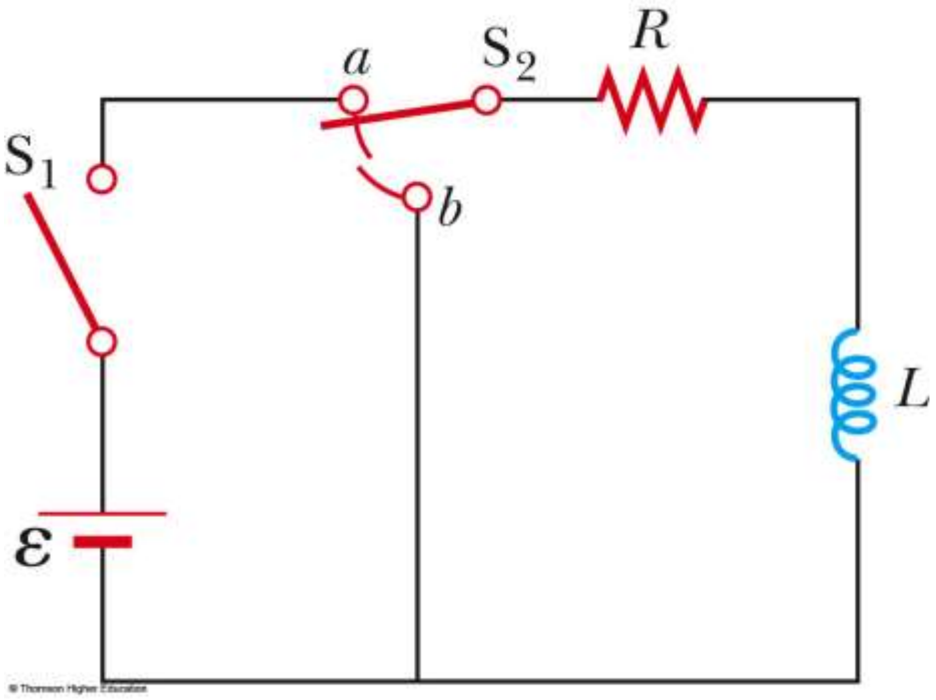
$$\frac{dI}{dt} = \frac{\varepsilon}{L} e^{-t/\tau}$$



RL Circuit Without A Battery

- Now set S_2 to position b
- The circuit now contains just the right hand loop
- The battery has been eliminated
- The expression for the current becomes

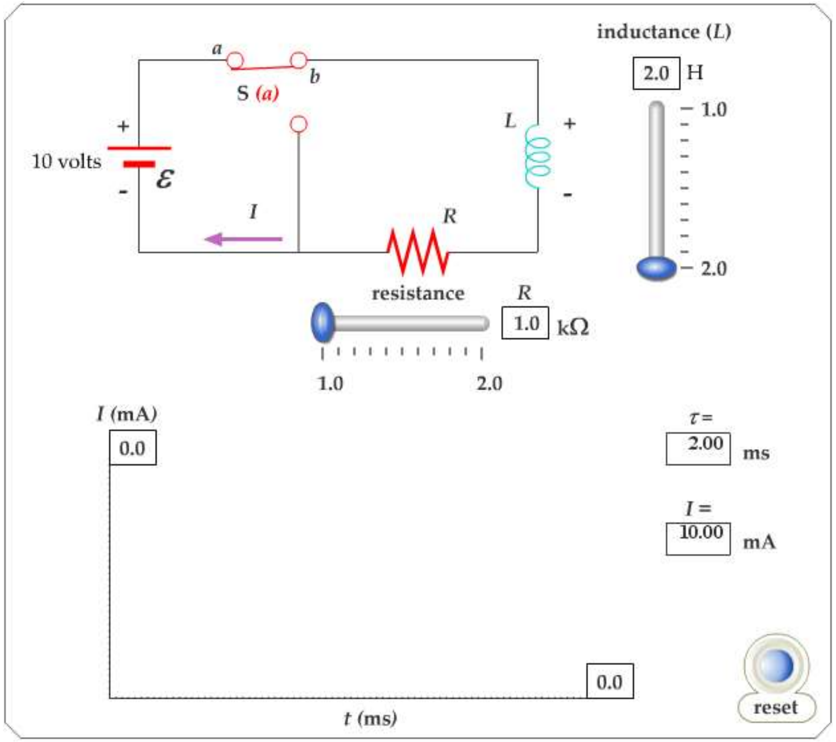
$$I = \frac{\varepsilon}{R} e^{-t/\tau} = I_i e^{-t/\tau}$$



PLAY ACTIVE FIGURE

Active Figure 32.2 (b)

Use the active figure to change the values of R and L and watch the result on the graph



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Energy in a Magnetic Field

- In a circuit with an inductor, the battery must supply more energy than in a circuit without an inductor
- Part of the energy supplied by the battery appears as internal energy in the resistor
- The remaining energy is stored in the magnetic field of the inductor

Energy in a Magnetic Field, cont.

- Looking at this energy (in terms of rate)

$$I\varepsilon = I^2 R + L I \frac{dI}{dt}$$

  - $I\varepsilon$ is the rate at which energy is being supplied by the battery
  - I^2R is the rate at which the energy is being delivered to the resistor
  - Therefore, LI (dI/dt) must be the rate at which the energy is being stored in the magnetic field

Energy in a Magnetic Field, final

- Let U denote the energy stored in the inductor at any time
- The rate at which the energy is stored is

$$\frac{dU}{dt} = LI\frac{dI}{dt}$$

- To find the total energy, integrate and

$$U = L\int_0^I I\ dI = \frac{1}{2}LI^2$$

Energy Density of a Magnetic Field

- Given $U = \frac{1}{2} L I^2$ and assume (for simplicity) a solenoid with $L = \mu_o n^2 V$

$$U = \frac{1}{2}\mu_o n^2 V\left(\frac{B}{\mu_o n}\right)^2 = \frac{B^2}{2\mu_o}V$$

- Since V is the volume of the solenoid, the magnetic energy density, u_B is

$$u_B = \frac{U}{V} = \frac{B^2}{2\mu_o}$$

- This applies to any region in which a magnetic field exists (not just the solenoid)

Energy Storage Summary

- A resistor, inductor and capacitor all store energy through different mechanisms
  - Charged capacitor
    - Stores energy as electric potential energy
  - Inductor
    - When it carries a current, stores energy as magnetic potential energy
  - Resistor
    - Energy delivered is transformed into internal energy

Example: The Coaxial Cable

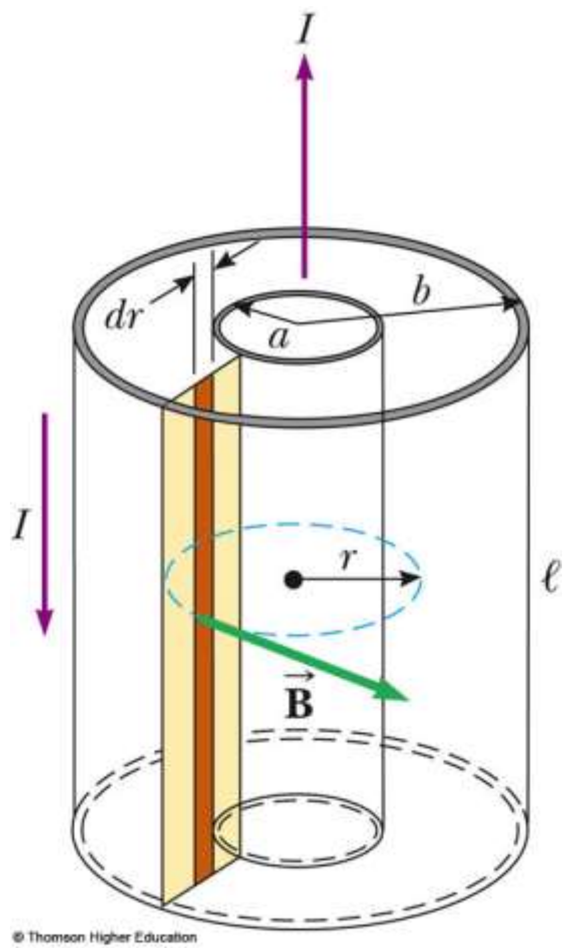
- Calculate L for the cable
- The total flux is

$$\Phi_B = \int B\, dA = \int_a^b \frac{\mu_o I}{2\pi r} \ell \; dr$$

$$= \frac{\mu_o I \ell}{2\pi} \ln\left(\frac{b}{a}\right)$$

- Therefore, L is

$$L = \frac{\Phi_B}{I} = \frac{\mu_o \ell}{2\pi} \ln\left(\frac{b}{a}\right)$$

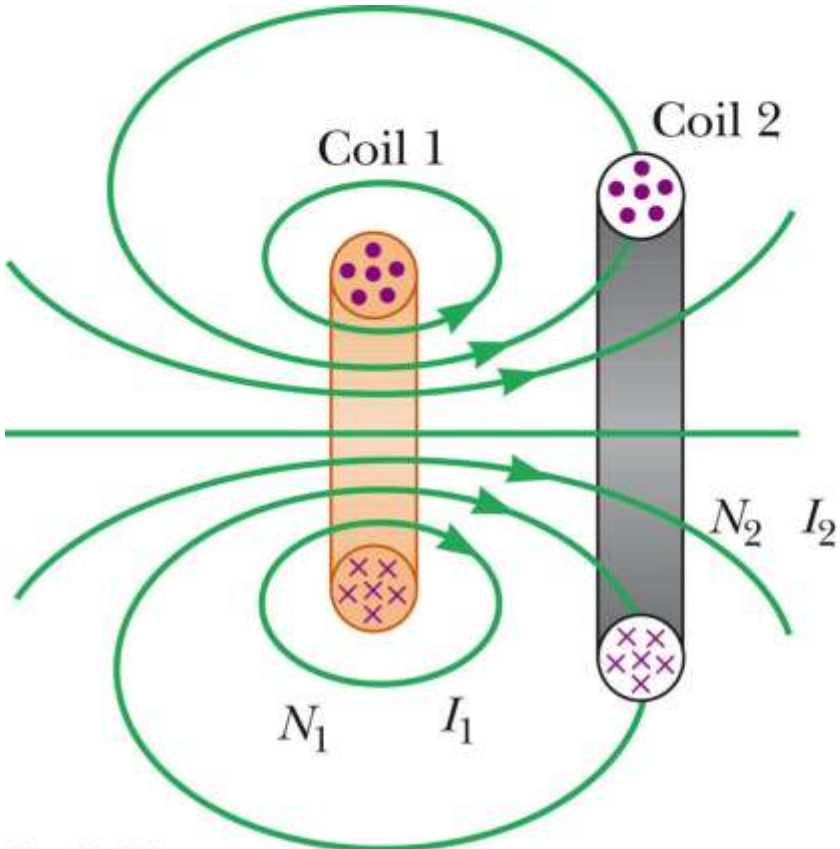


Mutual Inductance

- The magnetic flux through the area enclosed by a circuit often varies with time because of time-varying currents in nearby circuits
- This process is known as *mutual induction* because it depends on the interaction of two circuits

Mutual Inductance, 2

- The current in coil 1 sets up a magnetic field
- Some of the magnetic field lines pass through coil 2
- Coil 1 has a current I_1 and N_1 turns
- Coil 2 has N_2 turns



Mutual Inductance, 3

- The **mutual inductance** M_{12} of coil 2 with respect to coil 1 is

$$M_{12} \equiv \frac{N_2 \Phi_{12}}{I_1}$$

- Mutual inductance depends on the geometry of both circuits and on their orientation with respect to each other

Induced emf in Mutual Inductance

- If current I_1 varies with time, the emf induced by coil 1 in coil 2 is

$$\varepsilon_2 = -N_2 \frac{d\Phi_{12}}{dt} = -M_{12} \frac{dI_1}{dt}$$

- If the current is in coil 2, there is a mutual inductance M_{21}
- If current 2 varies with time, the emf induced by coil 2 in coil 1 is

$$\varepsilon_1 = -M_{21} \frac{dI_2}{dt}$$

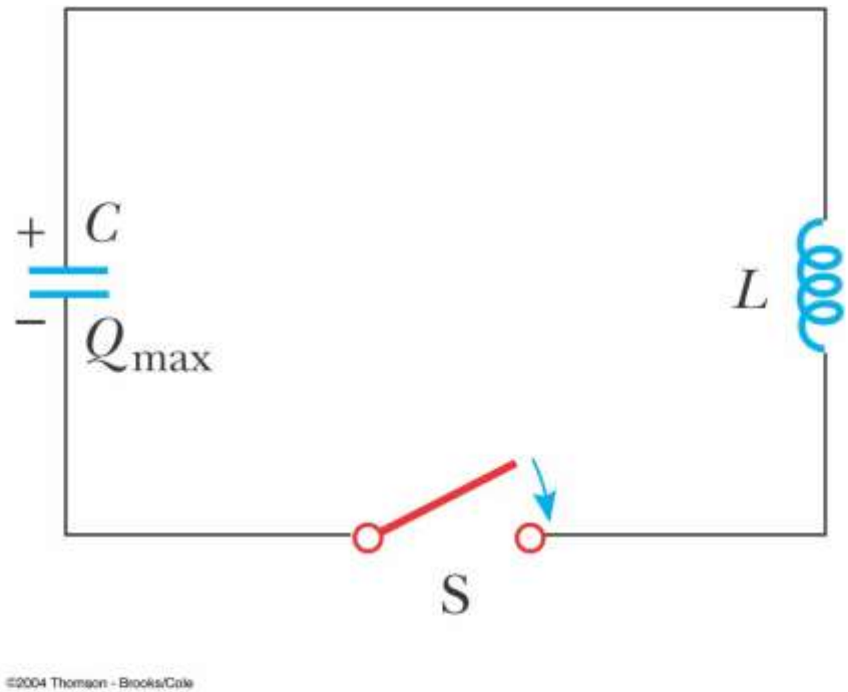
Mutual Inductance, Final

- In mutual induction, the emf induced in one coil is always proportional to the rate at which the current in the other coil is changing
- The mutual inductance in one coil is equal to the mutual inductance in the other coil
  - $M_{12} = M_{21} = M$
- The induced emf's can be expressed as

$$\varepsilon_1 = -M\frac{dI_2}{dt} \quad \text{and} \quad \varepsilon_2 = -M\frac{dI_1}{dt}$$

LC Circuits

- A capacitor is connected to an inductor in an LC circuit
- Assume the capacitor is initially charged and then the switch is closed
- Assume no resistance and no energy losses to radiation



Oscillations in an *LC* Circuit

- Under the previous conditions, the current in the circuit and the charge on the capacitor oscillate between maximum positive and negative values
- With zero resistance, no energy is transformed into internal energy
- Ideally, the oscillations in the circuit persist indefinitely
  - The idealizations are no resistance and no radiation

Oscillations in an *LC* Circuit, 2

- The capacitor is fully charged
  - The energy U in the circuit is stored in the electric field of the capacitor
  - The energy is equal to $Q^2_{max} / 2C$
  - The current in the circuit is zero
  - No energy is stored in the inductor
- The switch is closed

Oscillations in an *LC* Circuit, 3

- The current is equal to the rate at which the charge changes on the capacitor
  - As the capacitor discharges, the energy stored in the electric field decreases
  - Since there is now a current, some energy is stored in the magnetic field of the inductor
  - Energy is transferred from the electric field to the magnetic field

Oscillations in an *LC* Circuit, 4

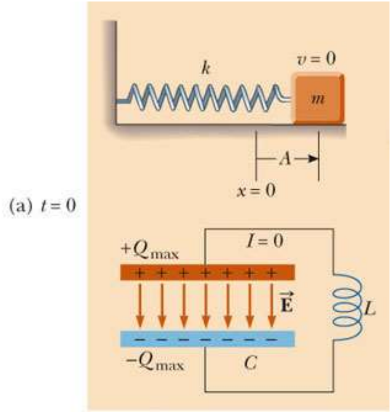
- Eventually, the capacitor becomes fully discharged
  - It stores no energy
  - All of the energy is stored in the magnetic field of the inductor
  - The current reaches its maximum value
- The current now decreases in magnitude, recharging the capacitor with its plates having opposite their initial polarity

Oscillations in an *LC* Circuit, final

- The capacitor becomes fully charged and the cycle repeats
- The energy continues to oscillate between the inductor and the capacitor
- The total energy stored in the LC circuit remains constant in time and equals

$$U = U_C + U_L = \frac{Q^2}{2C} + \frac{1}{2} L I^2$$

LC Circuit Analogy to Spring-Mass System, 1



- The potential energy ½kx^2 stored in the spring is analogous to the electric potential energy $(Q_{max})^2/(2C)$ stored in the capacitor
- All the energy is stored in the capacitor at t = 0
- This is analogous to the spring stretched to its amplitude

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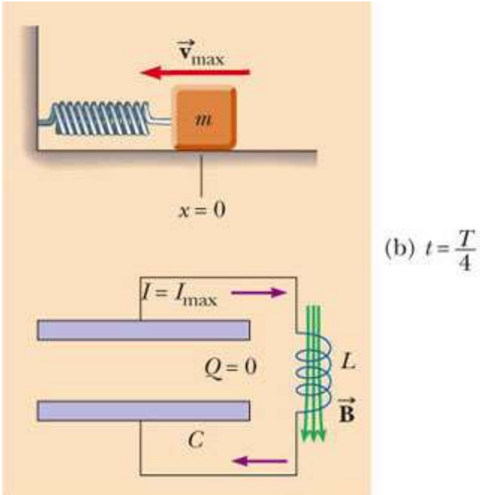
LC Circuit Analogy to Spring-Mass System, 2



- The kinetic energy (½ mv^2) of the spring is analogous to the magnetic energy (½ $L I^2$) stored in the inductor
- At t = ¼ T, all the energy is stored as magnetic energy in the inductor
- The maximum current occurs in the circuit
- This is analogous to the mass at equilibrium

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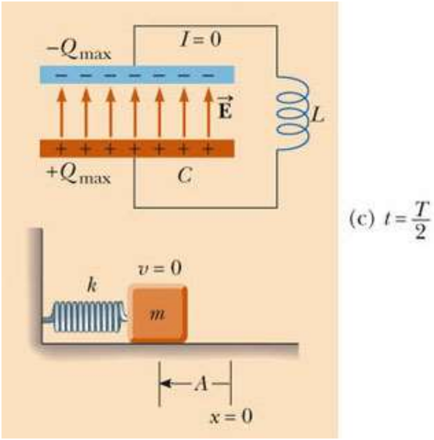
LC Circuit Analogy to Spring-Mass System, 3



- At t = ½ T, the energy in the circuit is completely stored in the capacitor
- The polarity of the capacitor is reversed
- This is analogous to the spring stretched to -A

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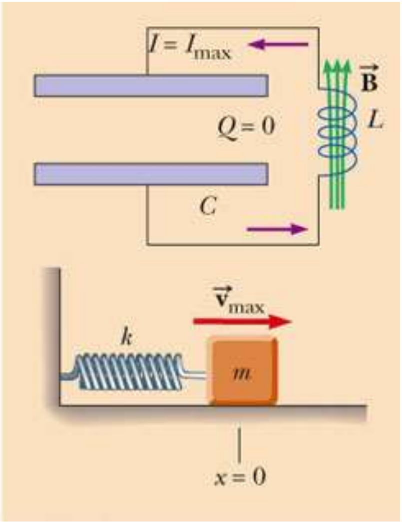
LC Circuit Analogy to Spring-Mass System, 4

(d) $t = \frac{3T}{4}$



- At t = ¾ T, the energy is again stored in the magnetic field of the inductor
- This is analogous to the mass again reaching the equilibrium position

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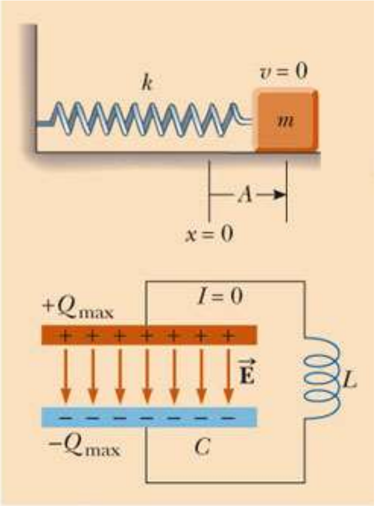
LC Circuit Analogy to Spring-Mass System, 5

(a) $t = 0$



- At $t = T$, the cycle is completed
- The conditions return to those identical to the initial conditions
- At other points in the cycle, energy is shared between the electric and magnetic fields

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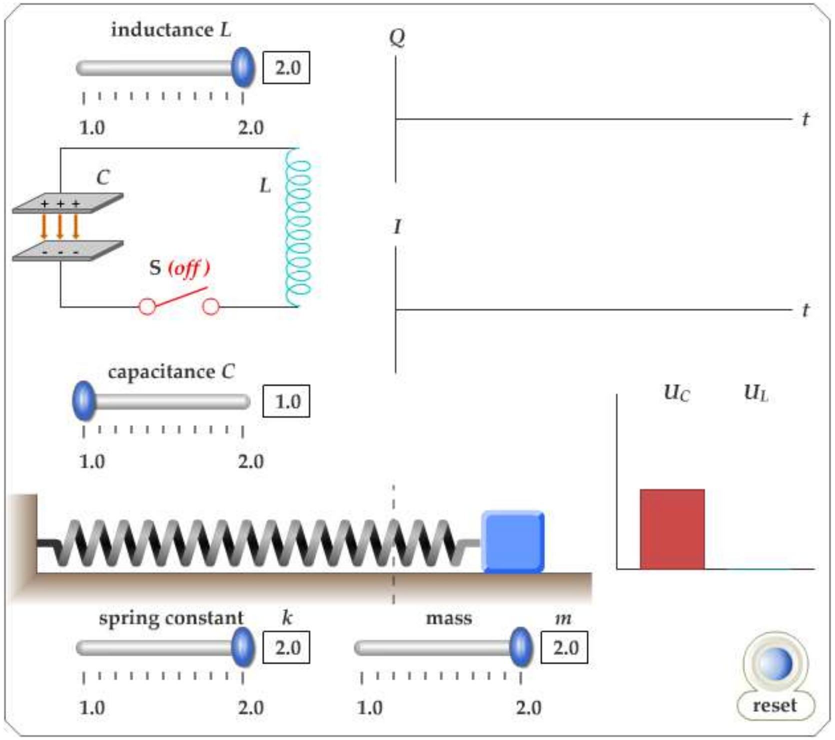
Active Figure 32.11

Use the active figure to adjust the values and L and C and see the effects on the current



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Time Functions of an *LC* Circuit

- In an LC circuit, charge can be expressed as a function of time
  - $Q = Q_{max} \cos(\omega t + \varphi)$
  - This is for an ideal LC circuit
- The angular frequency, ω, of the circuit depends on the inductance and the capacitance
  - It is the natural frequency of oscillation of the circuit

$$\omega = \frac{1}{\sqrt{LC}}$$

Time Functions of an *LC* Circuit, 2

- The current can be expressed as a function of time

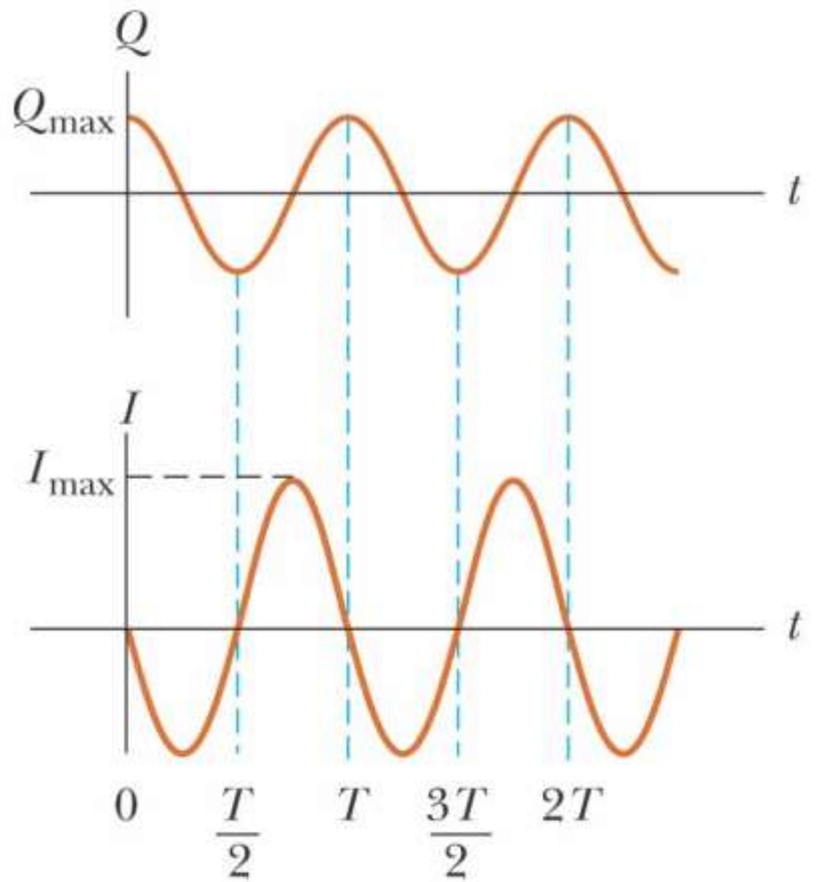
$$I = \frac{dQ}{dt} = -\omega Q_{max} \sin(\omega t + \varphi)$$

- The total energy can be expressed as a function of time

$$U = U_C + U_L = \frac{Q_{max}^2}{2c} \cos^2 \omega t + \frac{1}{2} L I_{max}^2 \sin^2 \omega t$$

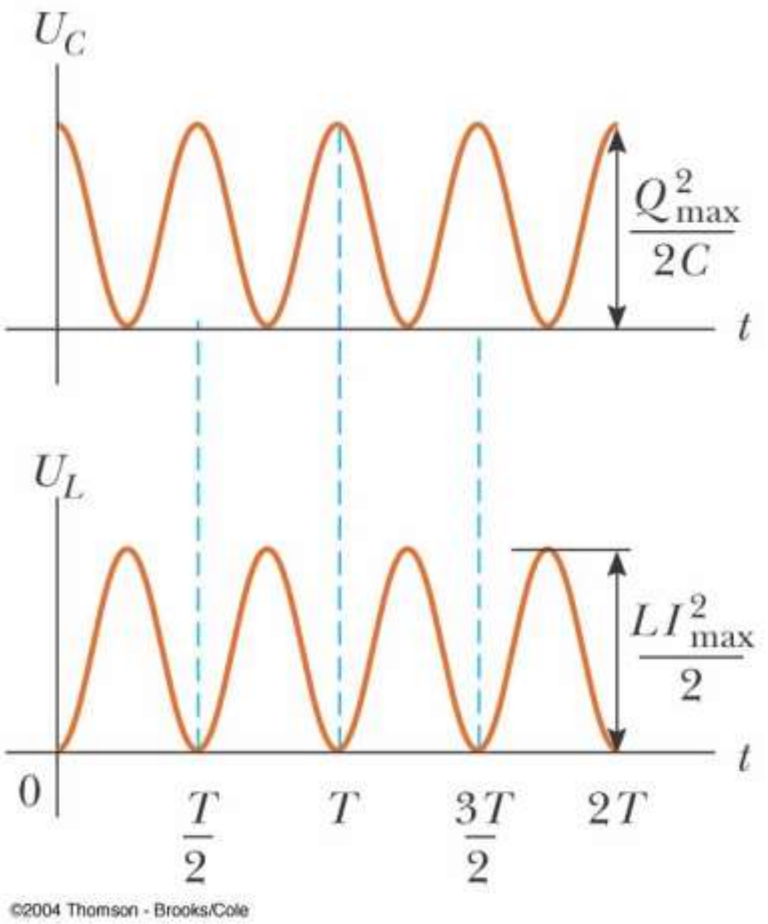
Charge and Current in an *LC* Circuit

- The charge on the capacitor oscillates between Q_{max} and $-Q_{max}$
- The current in the inductor oscillates between I_{max} and $-I_{max}$
- Q and I are 90º out of phase with each other
  - So when Q is a maximum, I is zero, etc.



Energy in an *LC* Circuit – Graphs

- The energy continually oscillates between the energy stored in the electric and magnetic fields
- When the total energy is stored in one field, the energy stored in the other field is zero

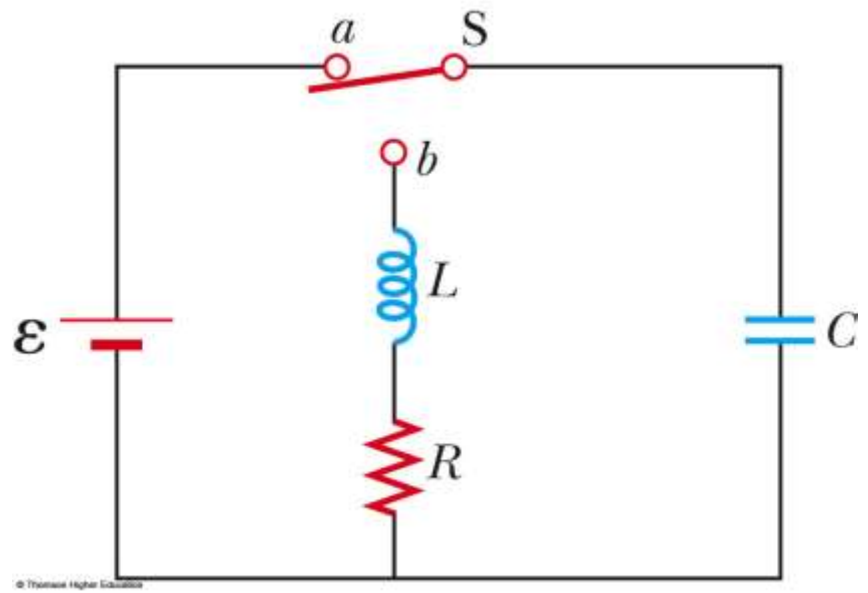


Notes About Real *LC* Circuits

- In actual circuits, there is always some resistance
- Therefore, there is some energy transformed to internal energy
- Radiation is also inevitable in this type of circuit
- The total energy in the circuit continuously decreases as a result of these processes

The *RLC* Circuit

- A circuit containing a resistor, an inductor and a capacitor is called an *RLC* Circuit
- Assume the resistor represents the total resistance of the circuit



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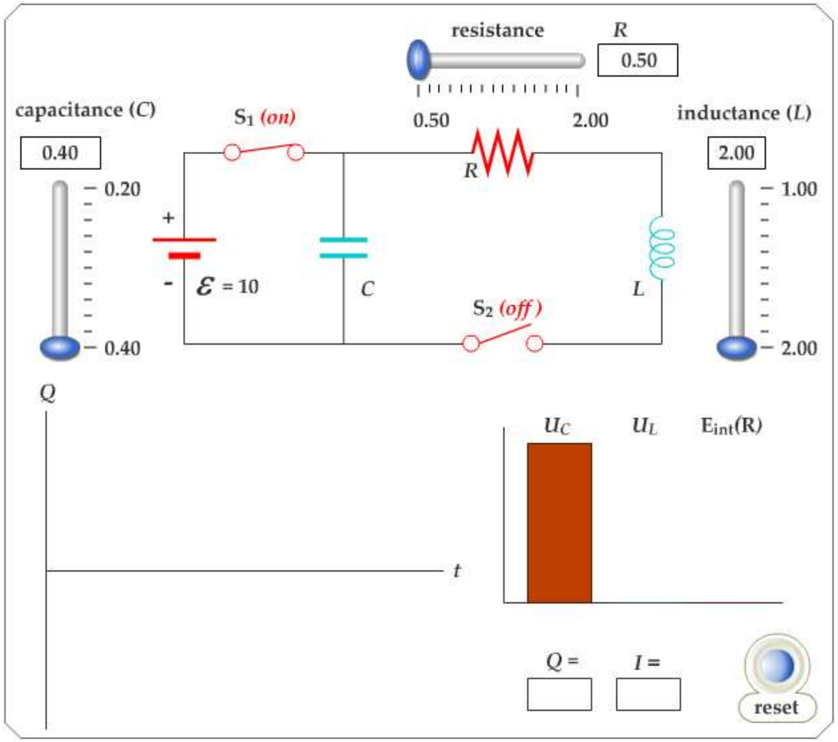
Active Figure 32.15

Use the active figure to adjust R, L, and C. Observe the effect on the charge



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RLC Circuit, Analysis

- The total energy is not constant, since there is a transformation to internal energy in the resistor at the rate of $dU/dt = -I^2R$
  - Radiation losses are still ignored
- The circuit's operation can be expressed as

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

RLC Circuit Compared to Damped Oscillators

- The *RLC* circuit is analogous to a damped harmonic oscillator
- When R = 0
  - The circuit reduces to an *LC* circuit and is equivalent to no damping in a mechanical oscillator

RLC Circuit Compared to Damped Oscillators, cont.

- When R is small:
  - The RLC circuit is analogous to light damping in a mechanical oscillator
  - $Q = Q_{max}\, e^{-Rt/2L} \cos \omega_d t$
  - ω_d is the angular frequency of oscillation for the circuit and

$$\omega_d = \left[\frac{1}{LC} - \left(\frac{R}{2L} \right)^2 \right]^{1/2}$$

RLC Circuit Compared to Damped Oscillators, final

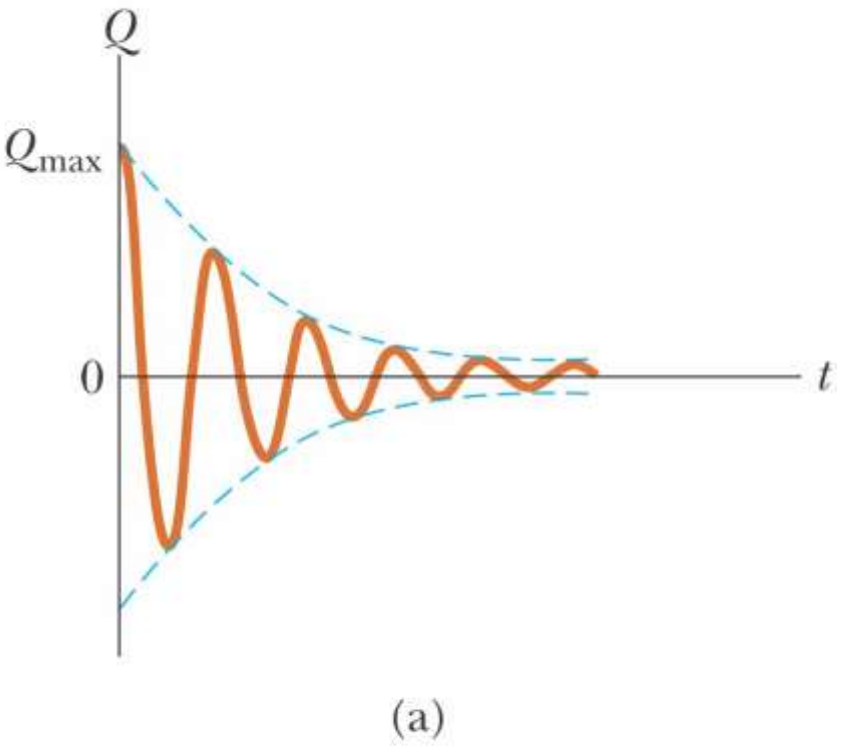
- When R is very large, the oscillations damp out very rapidly
- There is a critical value of R above which no oscillations occur

$$R_C = \sqrt{4L / C}$$

- If $R = R_C$, the circuit is said to be *critically damped*
- When $R > R_C$, the circuit is said to be *overdamped*

Damped *RLC* Circuit, Graph

- The maximum value of *Q* decreases after each oscillation
  - $R < R_C$
- This is analogous to the amplitude of a damped spring-mass system



Summary: Analogies Between Electrical and Mechanic Systems

TABLE 32.1

Analogies Between Electrical and Mechanical Systems

Electric Circuit		One-Dimensional Mechanical System
Charge	$Q \leftrightarrow x$	Position
Current	$I \leftrightarrow v_x$	Velocity
Potential difference	$\Delta V \leftrightarrow F_x$	Force
Resistance	$R \leftrightarrow b$	Viscous damping coefficient
Capacitance	$C \leftrightarrow 1/k$	(k = spring constant)
Inductance	$L \leftrightarrow m$	Mass
Current = time derivative of charge	$I = \frac{dQ}{dt} \leftrightarrow v_x = \frac{dx}{dt}$	Velocity = time derivative of position
Rate of change of current = second time derivative of charge	$\frac{dI}{dt} = \frac{d^2Q}{dt^2} \leftrightarrow a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$	Acceleration = second time derivative of position
Energy in inductor	$U_L = \frac{1}{2}LI^2 \leftrightarrow K = \frac{1}{2}mv^2$	Kinetic energy of moving object
Energy in capacitor	$U_C = \frac{1}{2}\frac{Q^2}{C} \leftrightarrow U = \frac{1}{2}kx^2$	Potential energy stored in a spring
Rate of energy loss due to resistance	$I^2R \leftrightarrow bv^2$	Rate of energy loss due to friction
RLC circuit	$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \leftrightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$	Damped object on a spring